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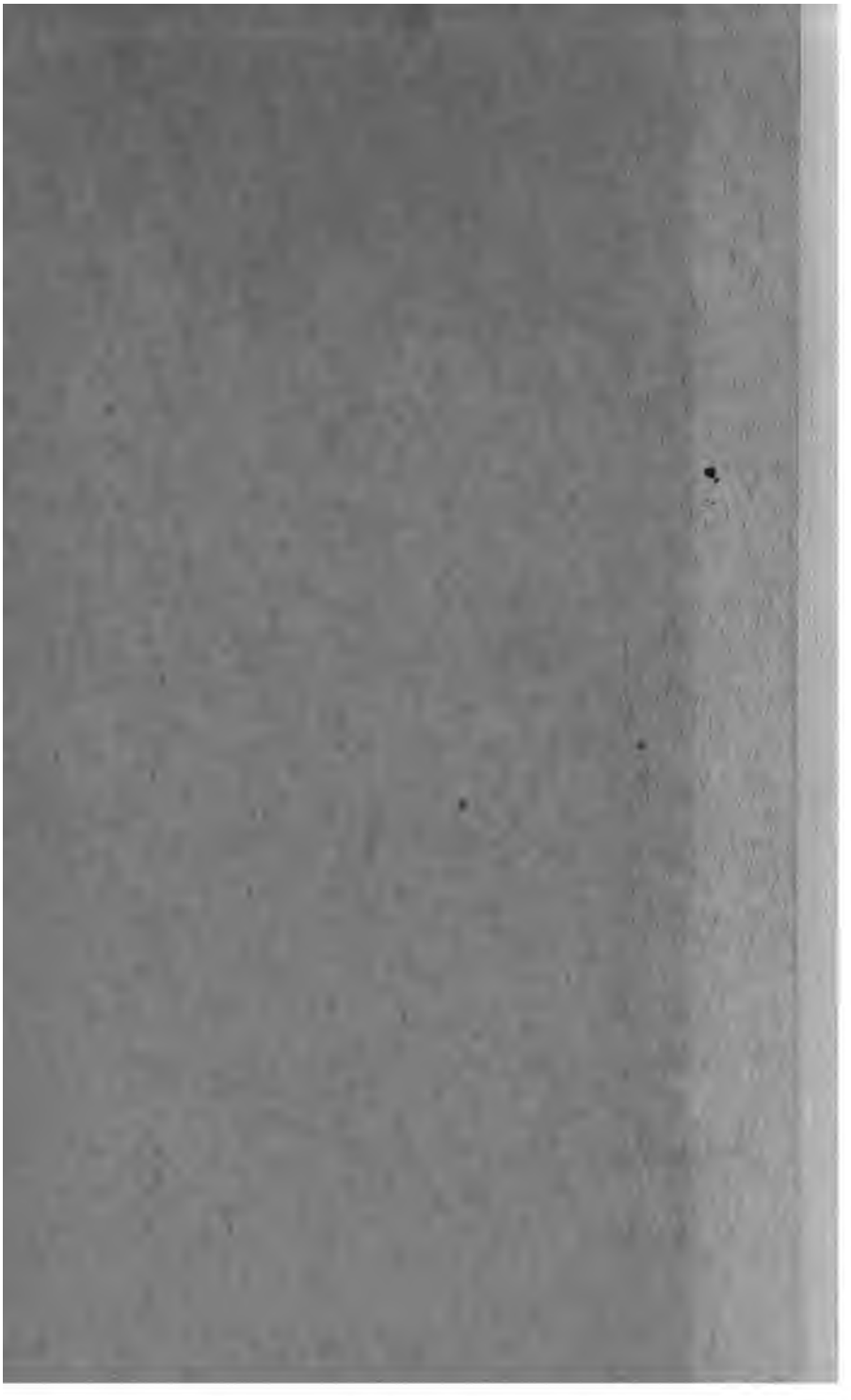

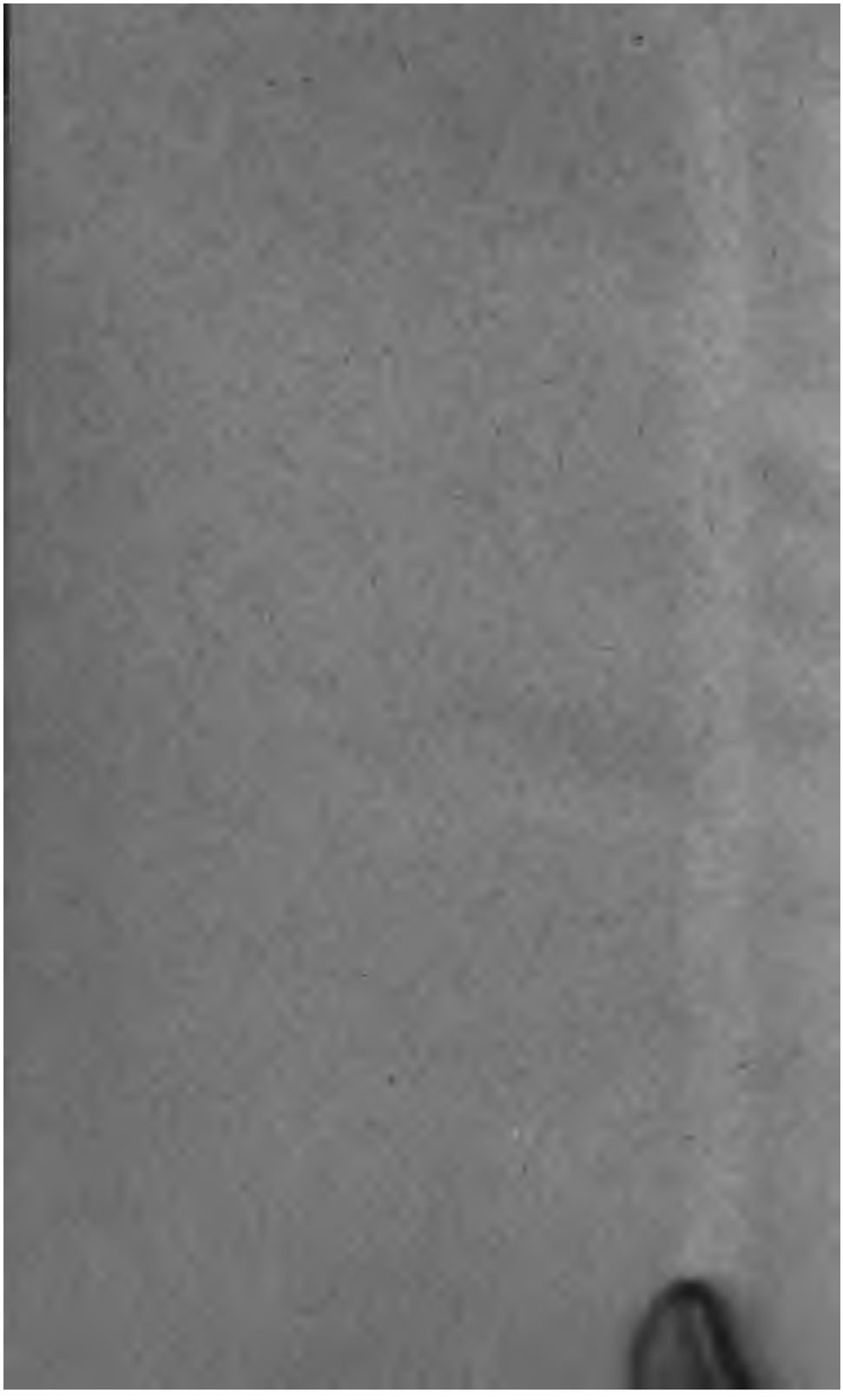

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THE FALSITY OF THE GRUBE METHOD

OF

TEACHING PRIMARY ARITHMETIC.

THESIS FOR THE DOCTORATE IN PEDAGOGY.

BY

SAUL BADANES.

NEW YORK, 1899.

SCIENCE DEPT.

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BIBLIOGRAPHY.

THE following books were consulted in the preparation of this thesis :

1. ACKERMAN, E. Pädagogische Fragen. Dresden, 1884.
2. Alternative, The. A Study in Psychology. London, 1882. (*Anonymous.*)
3. BAIN, A. Logic. New York, 1884.
4. CONDORCET. Moyens d'apprendre à compter surement et avec facilité. Paris, 1854.
5. CANTOR, M. Mathematische Beiträge zum Kulturleben der Völker. Halle, 1863.
6. COMTE, A. System of Positive Polity. London, 1876.
7. DUHAMEL. Des Méthodes dans les sciences de raisonnement. Paris, 1865.
8. Encyclopædia Metropolitana.
9. EGOROFF, Th. I. Metodeca Arifmetike. Moscow, 1893.
10. GRUBE, A. W. Leitfaden für das Rechnen in der Elementarschule. Berlin, 1881.
11. GOU, James. A Short History of Greek Mathematics. Cambridge 1884.
12. GALTON, F. Narrative of an Explorer in Tropical South Africa. London, 1853.
13. GOLDENBURG, A. E. Metodeca Natschalnoi Arifmetike. St. Petersburg, 1892.
14. HARTMAN, B., Dr. Der Rechenunterricht in der deutschen Volksschule. Leipzig, 1893.
15. HANKEL, HERMAN. Zur Geschichte der Mathematic im Alterthum und Mittelalter. Leipzig, 1874.

16. Hall, S. Contents of Children's Minds on Entering School. New York, 1894.
17. Husserl, E. C., Dr. Philosophie der Arithmetic. Halle a. S., 1891.
18. Herbart, J. F. Allgemeine Pädagogie und Pädagogische Schriften. Leipzig, 1875.
19. Hartman. Analyse des kindlichen Gedankenkreises. Annaberg, 1890.
20. Knilling, R. Zur Reform des Rechnenunterrichtes in der Volksschulen. München, 1884.
21. Kranke. Ausführliche Anleitung zu einem zweckmässigen Unterricht im Rechnen. Hannover, 1860.
22. Kehr. Geschichte der Methodic. Gotha, 1882.
23. Lewes. Problems of Life and Mind. New York, 1880.
24. Leroy, Georges C. The Intelligence and Perfectibility of Animals. London, 1870.
25. Lindsay. Mind in Lower Animals. New York, 1880.
26. Mill, J. S. System of Logic. New York, 1846.
27. Mill, J. Analysis of the Human Mind. London, 1878.
28. Preyer. Die Seele des Kindes. Berlin, 1890.
29. Pestalozzi's Sämtliche Werke. Brandenburg a. H., 1869–1872.
30. Rein, Dr., Pickel and Sheller. Das erste Schuljahr. Leipzig, 1892.
31. Romanes, J. Mental Evolution in Man. New York, 1889.
32. Sterner, Matthäus. Geschichte der Rechenkunst. München u. Leipzig, 1891.
33. Sachse, I. I. Der praktische, geistbildende und erzieliche Unterricht im Rechnen und in der Raumlehre. Osnabrück, 1887.
34. Taylor. Prehistoric Culture. New York, 1888.
35. Taine. On Intelligence. London, 1871.
36. Tanck, W. Das Rechnen auf der Unterstufe. Meldorf, 1884.
37. Unger. Die Methodic der practischen Arithmetic in historischen Entwickelung. Leipzig, 1888.
38. Ziller. Allgemeine Pädagogie und Grundlegung zur Lehre vom erziehenden Unterricht. Leipzig, 1884.

THE FALSITY OF THE GRUBE METHOD OF TEACHING PRIMARY ARITHMETIC.

CHAPTER I.

EXPOSITION OF THE GRUBE METHOD.

THE science of arithmetic was little developed with the ancients, because of their imperfect methods of notation, as is testified by their laborious processes of multiplication, and especially of division ; hence the modern problems of pedagogy that are related to the teaching of arithmetic could not have arisen then. The science of arithmetic received its greatest impetus when the decimal system of notation was introduced into Italy and France by Gerbert (Pope Sylvester II.), who was the first Christian priest to visit Seville and Cordova, the great centres of Mohammedan learning. With the introduction of the Hindoo-Arabic system the mind found great scope in being able to use large numbers easily. It also found no further need of oral arithmetic, and gave itself up entirely to mechanical processes, pursuing merely a practical end.

In examining all text-books in arithmetic written up to the middle of the eighteenth century we are struck with the mechanical ways of teaching the arithmetical operations. First, the rule was given, which the pupil was required to memorize ; then examples followed according to the rule. The numbers used in the examples were very large, the proof following each example. The interest of the learner was utterly ignored ; all his faculties were left dormant ; appeal was made merely to

memory, and consequently the latter was overtaxed. At the end of the eighteenth century, when all human institutions, all departments of human thought and human activity, were criticised with the object of placing them on a better basis, the need of popular education was felt for the first time, and with this need arose the demand for better methods and improved text-books suitable for the children of the masses. As a result of these demands text-books began to appear written with the purpose of making the subject-matter of instruction easier, clearer, and simpler for the learner. An excellent illustration of this period of history is to be found in the life and work of Marquis de Condorcet, a man of many-sided intellect and of a rich emotional nature. He was an indefatigable writer, was a member of the Convention that suggested reforms, and is especially interesting to us as a distinguished mathematician. As a member and secretary of the Academy of Sciences he wrote a tract on elementary arithmetic under the title "*Moyens d'apprendre à compter surement et avec facilité*"—(Means for Learning to Reckon Certainly and Easily). The contents of this little book perfectly justify its title. Its pages are full of observations from which teachers even at the present day may learn something. Condorcet's elementary arithmetic goes to prove that amid social reforms the need was felt of basing instruction on sound principles. But the spirit of the age was felt by many in different parts of Europe. It reached also Switzerland, and found there many enthusiasts for reform. Among these was John Henry Pestalozzi, who felt that he could render the highest usefulness to his fellow men by becoming a schoolmaster. It was he who gave its death-blow to the mechanical form of instruction that had preceded him.

He declared that instruction should be used as a means for the complete development of the power of the individual ; he also announced the principle that we should begin our instruction by giving the child an experience.

He said that observation, or sense-perception, is the first step of instruction, whether in form, number, or anything else. Since the principle of sense-perception has but a limited application to the study of arithmetic, and as this limit was unknown to him, Pestalozzi gave a

wrong turn to the study of arithmetic: this I hope to show when we come to criticise the method of Grube, who extended and amplified the principle of sense-perception.

Pestalozzi was the first to give prominence to oral arithmetic. He, however, went to the other extreme, by attaching very little value to written work, or to questions directly applicable to life.

His pupils barely knew the advantage of our decimal system of notation, but his name will always remain in the history of education as that of one who tried to make elementary arithmetic the common possession of all people.

At the end of his life he wrote that he was dissatisfied with his own method of teaching elementary arithmetic.

After Pestalozzi, the man that had the most influence on the method of teaching this subject was Grube.

In 1842 he published the first edition of his manual of arithmetic, under the title "Leitfaden für das Rechnen in der Elementarschule, nach den Grundsätzen einer heuristischen Methode; Ein pädagogischer Versuch zur Lösung der Frage: Wie wirkt der Unterricht sittliche Bildung?"

In the introduction to his manual Grube justifies the title of his book, because it deals with the moral influence of the School, and especially with the moral influence of instruction; further, he deals with the moral influence of instruction in elementary arithmetic. But in this his high aim of instruction he gave nothing new to the world; since Herbart—a mind of greater calibre—had declared, many years before Grube, that the aim of all instruction ought to be the formation of character.*

What, however, we wish to consider is his principles of instruction in elementary arithmetic, and also the deductions he made from his principles, as these have had, after Pestalozzi, the most influence on instruction in arithmetic.

Grube accepted the principle of Pestalozzi, that observation or sense, perception is the basis of all instruction: he extended it to the study of

---

* Grube wrote his manual one year after Herbart's death.

primary arithmetic, and by giving it a new turn formulated a principle, known since then as the monographic, or individual and all-sided, consideration of Number.

Numbers, according to Grube, are comprehended and recognized in the same manner as sensible objects ; hence the method of object-teaching is to be used also for teaching numbers. Further, every number, with all its component parts, from one to one hundred is to stand clear in the mind of the pupil ; every number is to form what is called in the school of Herbart a method whole, and is to be taught in the same way as a plant in Botany; and as only a thorough observation of a plant leads to a clear percept of the plant, in the same way only thorough observation of a number leads to a clear percept of the number. Hence the consideration of number ought to be all-sided ; all operations with numbers, and even the applied examples, are to serve merely to make clear the idea of an abstract number.

Grube argues for his principles of clear intuition of numbers, and of their monographic consideration, by saying that to separate the instruction of elementary arithmetic, according to the so-called four operations, is the same as to give children object-lessons of different things under the heading of one of their common qualities, such as size, form, color, etc., or to begin Botany with the system of Linnæus.

Just as a child never acquires the knowledge of an object by observing the same attribute in different objects, but by observing the different qualities of the same object ; and just as it is a mistake to show a beginner in Botany first the root of a plant, then the stalk, and later the leaves, since he should first see the whole plant, and only afterwards the parts,—so it is a mistake to teach the child to-day $2 + 2 = 4$, and after several weeks to turn to the subtraction of $4 - 2 = 2$, etc., as by this method the child is not led to comprehend the number 4.

To acquire a thorough knowledge of the number 4, the child must learn, simultaneously with the knowledge of $2 + 2 = 4$, that $4 - 2 = 2$, $2 \times 2 = 4$, $4 \div 2 = 2$.

The separation of the study of numbers according to the four operations, argues Grube, weakens the perception (intuition) of number ; hence to strengthen the intuition of a number we are to exhaust all

possible operations within the limits of one number before we proceed to the next following number. Further, he argues that only the numbers from one to one hundred are directly intuitable; and as all reckoning with larger numbers can be performed by relating them to the first hundred, there is good reason why we should endeavor to have each number with all its constituent parts in the series from one to one hundred stand clearly before the soul of the pupil.

Further, Grube says that out of the all-sided intuition of each individual number the four fundamental operations will follow as a direct result.

Grube's course of study covers four years, four hours a week.

First year. The study of numbers from 1 to 10.

Second year. The study of numbers from 10 to 100.

Third year. The study of numbers from 100 to 1000 for the first half of the year; and for the second half of the year the study of any number.

Separate exercises for each operation.

Fourth year. Fractions. The first half of the year, all-sided observation of fractions; the second half of the year, practice in the four operations of fractions.

As the best way of judging a method is by seeing it in operation, let us present the way Grube teaches the number 4 and the number 6. Each number is to be taught under the following set form:

I. Pure *number:*

(*a*) Measuring (that is to say, separating or resolving each number into equal and unequal parts) and comparing;

(*b*) Rapid work;

(*c*) Combining.

II. Applied number.

The symbols of numbers and of operations Grube introduces with the first lesson, and with the study of each number its symbol.

The study of the number 4 is called by Grube the Fourth Step.

FOURTH STEP.

THE FOUR.

I. The Pure Number—Measuring and Comparing. (To be veloped by the question-and-answer method.)

(*a*) Measuring with 1:

				Four lines are used as an objective illustration of numb
|			1	$1+1+1+1=4$. $(1+1=2, 2+1=3, 3+1=$
|			1	$4 \times 1 = 4$
|			1	$4-1-1-1=1$
|			1	$1 \div 4 = 4$

(*b*) Measuring with 2:

|	|	2	$2+2=4$
|	|	2	$2 \times 2 = 4$
				$4-2=2$
				$2 \div 4 = 2$

(Grube writes the dividend on the right-hand side of the divisor.)

(*c*) Measuring with 3:

|	|	|	$3+1=4$; $1+3=4$
				$1 \times 3 + 1 = 4$
				$4-3=1$; $4-1=3$
				$3 \div 4 = 1$ (1) 3 in 4, once and 1 remainder.

Name animals with 4 legs and with 2 legs.

Wagons and vehicles with 1 wheel; 2 wheels and 4 wheels.

4 is how many more than 3; 2 more than 2; 3 more than 1?

3 is 1 less than 4; 1 more than 2; 2 more than 1.

1 is 3 less than 4; 2 less than 3; 1 less than 2.

4 is four times 1; twice 2.

1 is the fourth part of 4; 2 one half of 4.

Of what equal and unequal numbers can we form the number 4?

RAPID WORK.

$2 \times 2 - 3 + 2 \times 1 + 1 - 2$ doubled $=$?

$4-1-1-1+1-3$ is how many less than 4?

COMBINING.

What number must I take twice in order to get 4?

4 is twice what number?

Of what number is 2 the half?

Of what number is 1 the fourth?

What number can we take twice away from 4?

What number is 3 greater than 1?

How much must I add to the ½ of 4 in order to get 4?

How many times 1 is the half of 4 less than 3?

APPLIED NUMBERS.

Caroline had 4 tulips in her vase, which she neglected to water: one wilted; then another; then another. How many had she left?

How many cents in a 2-cent piece?

How many cakes can you buy for 4 cents, if each costs 1 cent? When each costs 2 cents?

If a top costs 2 cents, how much will two tops cost?

John paid for two cakes a 3-cent and a 1-cent piece; what was the cost of each cake?

One quart has 2 pints; how many pints in 2 quarts?

What part of 4 cents is 1 cent?

What part of 4 cents is 2 cents?

The above set form is to be followed in the study of each number from one to one hundred, the number 1 not being excepted.

When Grube reaches the Sixth Step, which is a study of the number 6, he says that the pupil has made such progress by his thorough study of the first five numbers, that he will be able to write out for himself that part of the set form called Measuring and Comparing.

Grube continues to say that the teacher has only to ask the pupil what he knows of the number 6, and the latter will write out the following table after the set form already pursued.

THE SIX.

The teacher says: "*Compare* 6 *with* 1."

And the pupils write:

$$1+1+1+1+1+1=6$$
$$6\times 1=6$$
$$6-1-1-1-1-1-1=1$$
$$1\div 6=6$$

"*Compare with* 2 :"

$$2+2+2=6$$
$$3\times 2=6$$
$$6-2-2=2$$
$$2\div 6=3$$

"*Compare with* 3:"

$$3+3=6$$
$$2\times 3=6$$
$$6-3=3$$
$$3\div 6=2$$

"*Compare with* 4:"

$$4+2=6$$
$$1\times 4+2=6$$
$$6-4=2$$
$$4\div 6=1\ (2)$$

"*Compare with* 5 :"

$$5+1=6$$
$$1\times 5+1=6$$
$$6-5=1$$
$$5\div 6=1\ (1)$$

"Compare the number of legs of different animals of 6 leg quadrupeds and bipeds."

Further, the pupil is to write out the following table by review :

$6 = 5 + 1$ (is one more than 5) ; $4 + 2$; $3 + 3$; $2 + 4$; $1 +$

$5 = 6 - 1$ (is one less than 6) ; $4 + 1$; $3 + 2$; $2 + 3$; $1 +$

$4 = 6 - 2, 5 - 1, 3 + 1, 2 + 2, 1 + 3.$

$3 = 6 - 3, 5 - 2, 4 - 1, 2 + 1, 1 + 2.$

$1 = 6 - 5, 5 - 4, 4 - 3, 3 - 2, 2 - 1.$

6×1 (is 6 times 1), $3 \times 2, 2 \times 3.$

$3 = \frac{1}{2} \times 6$ (is one half of 6).

$2 = \frac{1}{3} \times 6$.

$1 = \frac{1}{6} \times 6$.

"Of what three like numbers is 6 composed?"

"Of what three unlike numbers?"

RAPID WORK.

$1 \times 2 + 1 \times 2 + 1 \times 2 - 1 \times 2 \times 3 - 5 \times 5 = ?$

I have 6 cents, and spend 1 cent, 2 cents, and 3 cents; how many nts have I left?

$4 + 2 - 3$ is how much less than 6?

COMBINING.

What number is 3 times 2?

What number is twice 3?

What number can you take three times from 6 and twice from 4?

How many times 1 is a half of 6 more than a half of 4, and how uch less than 5?

If I take a number twice away from 6 and have 2 left, what is the mber?

How many times is $\frac{1}{3}$ of 6 contained in 4?

The half of 4 equals what part of 6?

II. Applied Number.

How many times 1 cent, 2 cents, and 3 cents is 6 cents?

How many quarts is 6 pints?

What will 3 litres of milk cost at 2 cents a litre?

3 tops cost 6 cents; what is the cost of 1?

What is the cost of 3 sheets of paper at 2 cents a sheet?

I have 6 apples in 3 pockets; how many apples in each pocket?

CHAPTER II.

A PSYCHOLOGICAL ANALYSIS OF THE CONCEPT OF NUMBER.

THE principles of the Grube Method have found many adherents in America, some adopting the method in its entirety, some appropriating only special features of the method. These principles have reached such a wide popularity that in many states we find teachers, at their examination for license, asked to explain the Grube Method and its advantages; and teachers in Normal Schools are found among its stanch supporters.

Because of its wide acceptance, it is opportune to examine the basis of the Grube Method, to note its advantages and to point out its many disadvantages. But before we direct our criticism to the method, we shall have to examine a question of fundamental importance; namely, the question of the origin and nature of the idea of number.

Until recently this question has had interest only for the mathematician and the metaphysician, but it ought to attract the attention of the teacher; for his view of the nature of number must have a decisive influence upon the method of instruction in primary arithmetic.

What then is the origin of number?

Believing that the metaphysical explanation of number cannot render a service to Pedagogy, we shall limit ourselves merely to a brief description and explanation of the origin of number as offered by the psychology of individual minds.

According to this science, the most fundamental condition of all our psychical states is the experiencing of some transition or change; without change there can be no consciousness; hence the most fundamental property of consciousness is the experience of difference or discrimination. Every difference in consciousness has its mark by which it is distinguished; however, we feel the difference only in degree or quantity, and it is not until later that we learn to localize our sensations.

Though localization is an important characteristic of our sensation, it is not a primary characteristic, but a result of slow acquisition, as testified by experimental psychology and by observation on infants; hence, the only characteristic that is primary in the act of discrimination has reference to that of degree or quantity.

Of two differing sensations of light, one is felt to be more intense than the other; it is the same with pleasure and pain, and with psychical states of every description. The attribute of quantity is equally inseparable from objective as well as subjective facts.

Further, psychology tells us that every difference has its correlative resemblance; neither is possible without reflecting the other; no sense of difference can therefore arise without a relative resemblance from which it is discerned. With the part, the whole is given; with the individual, the class is given to which it belongs. In the same manner we discriminate — at least implicitly, the unity and plurality of objects.

The perception of "two" children usually reach without the help of their parents, from whom they only receive the word "two;" the number "three" is reached by children in a similar way. How higher numbers are developed will be shown later.

For the purpose in hand, the question of the growth and development of the idea of number is of greater importance than the question of origin.

As the idea of number is a social product, and was developed by the co-operation of successive generations of the human race, it is in human history that we may hope to find exhibited all the stages of the growth of the idea of number. Of course we need not adhere in our exposition strictly to the historical order in all its details; this would introduce here a great deal of material unnecessary for our purpose.

Let us see, then, what the history of the human race can tell us of the growth of the idea of number. In studying the development of the idea of number, we shall have to divide its genesis into three principal periods or stages — periods which illustrate a prominent characteristic in the process of the growth of this idea, and, at the

same time, suggest all the important stages of the individual development of the human race. The three periods are the following :

1. The period of spontaneous numbers.

2. The period when images were introduced which facilitate the ' extension of our numeration beyond the spontaneous stage. With this period the inductive process of reasoning began in the human race.

3. The period of the formation of numerical concepts and judgments. This was the period when signs were substituted for images —an indication of the appearance of the deductive process of reasoning ; with it, arithmetic began to develop as a science.

The spontaneous period which precedes conscious counting is characterized by the fact that we are able to form an exact estimate of visual or tactual objects, or an auditory estimate of tones, by mere comparison of sense impressions.

In the study of primitive races, of very young children, and even of animals, we can find illustrations of this period. We may cite as illustrations the following three well-known examples, found in the writings of F. Galton, C. Leroy, and Preyer.

Francis Galton,* in describing the obtuseness of the Damaras, tells us the following : " In practice, whatever they may possess in their language, they certainly use no numeral greater than 3. When they wish to express 4 they take to their fingers. . . . They puzzled very much after 5, because no spare hand remains to grasp and secure the fingers that are required for " units; " yet they seldom lose oxen. The way in which they discover the loss of one is, not by the number of the herd being diminished, but by the absence of a face they know.

" When bartering is going on, each sheep must be paid for separately : thus, suppose two sticks of tobacco to be the rate of exchange for one sheep; it would sorely puzzle a Damara to take two sheep and give him four sticks. I have done so and seen the man put two of his sticks apart and take a sight over them and at the

* " Narrative of an Explorer in Tropical South Africa," p. 133.

sheep he was about to sell. Having satisfied himself that he was honestly paid, and finding to his surprise that exactly two sticks remained in hand to settle the account for the other sheep, he would be afflicted with doubts; the transaction seemed to come out too pat to be correct, and he would refer back to the first couple of sticks; and then his mind got hazy and confused, and wandered from one sheep to the other; and he broke off the transaction, till two sticks were put into his hand and one sheep driven away, and then the other sticks given him and the second sheep driven away."

Charles Leroy, a well-known naturalist, whose life-work was that of a ranger at Versailles, brings, as an illustration of the experience of the idea of number among animals, the following case:

"Among the various ideas which necessity adds to the experience of animals, that of number must not be overlooked. They count, that is certain; and though, up to the present time, their arithmetic appears weak, perhaps it may be possible to strengthen it.

"In those countries in which game is much preserved, crows are made war upon, because they take away the eggs, and destroy the hopes of the laying season. The nests of these destructive birds are carefully noticed, and to destroy the voracious family at a blow they seek to kill the mother while sitting.

"Among the parent birds some are very suspicious, and desert their nests as soon as any one approaches them. To lull suspicion, a carefully covered watch-house is made at the foot of the tree in which there is a nest, and a man conceals himself in it to await the return of the parent bird; but he waits in vain if she has ever before been shot at in the same manner. She knows that fire will issue from the cave into which she saw a man enter. While maternal love keeps her eye fixed upon her nest, fear prevents her return till night hides her from the sportsman's sight.

"To deceive this suspicious bird, the plan was hit upon of sending two men to the watch-house, one of whom passed on, while the other remained; but the crow counted, and kept her distance.

"The next day three went, and again she perceived that only two returned. In fine, it was found necessary to send five or six men to the

watch-house to put her out in her calculation. The crow, thinking that this number of men had but passed by, lost no time in returning.

"This phenomenon, always repeated when the attempt is made, is to be recorded among the very commonest instances of the sagacity of animals."

Of course, we must not suppose that the bird counted the men by any process of notation, but the contrast between the simultaneous sense-perceptions each time was sufficient for the spontaneous inferences.

Preyer tells of a child who, at the age of a year and a half, could tell at a glance whether one of her ten animals was missing or not.* (Of course, such a case of fine discrimination is very rare.)

We also find him telling the following story of his own child: "In his eighteenth month, after having been accustomed to bring to his mother two towels, which he would afterwards carry back to their place, on one occasion had only one towel given back to him; he, with inquiring look came back to get the second."

Professor Preyer, in his paper on Arithmogenesis, read before the International Congress of Experimental Psychology, held in London in 1892, confirms the fact that there is such a spontaneous stage of numbers, and further brings arguments and evidence that unconscious numerical discriminations take their origin in our auditory sense.

In this period, we cannot carry distinct numeration beyond the first four numbers; it is true that we can form estimates a good deal beyond that point by skilfully grouping objects: then, of course, form comes to our aid; further, students in psychological laboratories can estimate a great deal beyond that point, as a result of an educated discrimination.

The second stage of the genesis of the idea of number began with the discernment of the first non-spontaneously developed number; the condition of this discernment being the notice of an addition of a unit to the highest spontaneously developed number. This discernment was made permanent by indicating it by an image, which represented not only a sum or a plural quantity, but also the process of obtaining it from the next lower number; hence, it represented a definite number.

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* W. Preyer, "The Mind of the Child," Part II., p. 8.

When a primitive savage met with a sum or plural quantity which he was interested to determine, it would naturally occur to him to withdraw from the sum a group of the highest spontaneously developed numbers and then withdraw a unit from the remainder of the sum and add it to the group, indicating the newly formed group by an image; he would keep on withdrawing each time a unit from the remainder and constantly adding it to the newly formed image, until all the units of the plural quantity were exhausted.

Later on, we find savages using the same image for like numbers. In this way counting was invented.

The Indians of the Tamanaca tribe count on their fingers. The numerical word one signifies with them one finger; the numerical word five signifies the whole hand; six, one on the other hand; seven, two on the other hand.

The Caraibs at the Orinoco River have separate names for the numbers from 1 to 4; five is called four and the other; six is called the hand and one more; seven, the hand and two more.

Thus the object of counting is seen to be the determination of a plural quantity which cannot be spontaneously grasped. Its process consists in forming numbers by the addition of a unit to the number next below in magnitude, and thus arranging numbers in a scale of which each higher degree exceeds by a unit the sum of the preceding degree.

With every addition of a unit in this series, we make a new inference, and as this inference is drawn in connection with our handling of sensible objects, and concerning them, it is therefore inductive in its nature.

The formation of images of the successive numbers is necessary to these inferences in order to fix our attention and give coherence to our reflections.

For the representation of numbers, pebbles, fine nuts, beans, knots tied on a string, and the fingers were first used. With the extension of the process of counting, the number of images began to multiply; this produced a confusion in the mind of the primitive races and made it impossible for them to retain the images. To obviate this difficulty, they hit upon the method of making a few images denote a multitude

of numbers. The secret of this plan lay in grouping or regarding certain numbers as individual things. For instance, having advanced in counting up to 5 or 10, they indicated the new group of 5 or 10 by a special image and then they continued to add to this group the images of preceding numbers of this group.

In a similar manner, when five fives or ten tens were reached, they denoted the new group by an image and proceeded to add to this new group with its images preceding numbers with their images.

Thus, in this period was laid the basis of notation, which is the necessary complement of numeration, and possibly also the basis of the decimal system of notation, as suggested by the fingers of both hands.

But the process of forming numbers by mere addition of a unit to the number next below in the scale could only be extended to the formation of small numbers, as each new judgment depended upon counting concrete things; hence it was necessary to find other methods which should help in forming large numbers, and thus shorten the process of counting.

This demand was completely satisfied in the third period of the development of numbers. In this period the mind became utterly detached from sensible observations; by the process of reasoning new numbers were formed from the numbers acquired in the second period, that is to say, by passing from judgments retained in memory to new judgments and conclusions.

Knowing from the previous (second) period that $7+1+1+1+1+1=12$; also that $1+1+1+1+1=5$, the conclusion reached was that $7+5=12$. This process of reasoning, which is a deductive process, when fully set forth in its logical form would give the following:

	Seven $+1+1+1+1+1=12$
But	$1+1+1+1+1=5$
Hence	$7+5=12$

Of course, in the actual psychological process, the logical order was not followed. This is required only for the purpose of proof; that is to say, in our ordinary deductions we do not start with two antecedently known judgments, from which we draw a third judgment as a

conclusion; but the conclusion is the first judgment that distinctly presents itself to the mind.

The method of this stage, besides offering many ways of forming numbers additively, also suggests the process of forming numbers subtractively—thus: if $7 + 5 = 12$, $12 - 5 = 7$, and $12 - 7 = 5$. Considerations of such nature have very likely led to the subtractive form of IX, XIX, etc. The number 19 was probably more readily formed from the concept 20 than from the concept 10.

In the efforts to form numbers deductively, it became indispensable to substitute artificial signs for the images of the previous period, as these images embarrassed the efforts at deductive reasoning.

In these images we find a suggestion of likeness between the sign and the fact signified, but the artificial signs offer no resemblance to the numbers themselves; hence they facilitate the reasoning process, especially when abstraction is needed.

The formation of numbers deductively gave rise to the beginning of elementary calculations. These consisted of performing operations upon signs that stood for the direct or indirect result of counting.

The first elementary calculations were limited to addition, subtraction, and to the abbreviated form of addition, i.e., multiplication; division was developed at a later period, owing to the abrupt increase of complexity in mathematical reasoning.

To this period belongs the invention of the method whereby a small number of signs could represent all possible numbers. This method is an artifice and consists in indicating the value of numerical units by the position of their digits.

In addition to this fundamental artifice, another improvement was made—the invention of the multiplication table—by the help of which all products are reducible to the first nine numbers.

With these two improvements, that facilitate greatly our written calculations, arithmetic as a science began, and numbers began to be formed in a great many ways, as, in every operation of arithmetic, we form the required number by the help of the given number.

Here ends our survey of the growth of the idea of number; but before dismissing this topic we wish to add the following: that in the

actual growth of the idea of number there is no separate demarcation between one period and the next; but, on the contrary, one period overlaps the other, one period is interwoven with the other. All our psychical states and processes are synthetical in their nature, yet they appear to be analytical while we study them.

It is especially important to bear in mind the synthetical nature of our psychical processes, in order to understand the second stage of the growth of the idea of number. This stage occupies an intermediate position between the sphere of implicit and explicit mental activity partaking of the characteristic qualities of each.

The images of this period, whether they are sights or sounds, recall our sensations with a diminished energy, and at the same time, like arbitrary symbols, they represent definite numbers; but they differ from the symbols in not being general in their nature; hence they do not permit of combination and permutation in an infinite number of ways. Images have a significance only for direct and immediate purposes; they can facilitate the rise of the inductive process, but for mediate and deductive processes the use of arbitrary symbols are indispensable.

CHAPTER III.

A CRITICISM OF THE GRUBE METHOD.

We are now ready to enter upon the critical examination of the Grube Method, and shall begin naturally with the consideration of its fundamental principles. Grube came to his fundamental principle, that numbers are comprehended and recognized in the same way as sensible objects, and to the deductions he made from it, not by a thorough psychological analysis of the idea of number, but by the method of analogy, in the following way:

We cannot have a knowledge of an object by observing it from one point of view or by perceiving one of its qualities; hence we cannot perceive a number by learning to-day addition, and after several weeks subtraction, and later on multiplication, etc.

Further, in object-lessons, we are to teach one object at a time, to consider all its qualities, and to proceed to the study of another object only after a thorough study of the first; hence, concludes Grube, we are to teach one number at a time. We are to have an all-sided consideration of each individual number, and we are to exhaust all possible operations within the limits of one number, before we proceed to the next. And lastly, a thorough study of an object consists of four steps:

1. Observation of the entire object.
2. Separating of its parts and comparing them.
3. Combining all the parts.
4. The use and purpose of the object.

Hence each number is to be considered under the following set form:

I. Pure number:

- (*a*) Measuring (that is to say, separating or resolving each number into equal and unequal parts) and comparing;
- (*b*) Rapid work;
- (*c*) Combining.

II. Applied number.

As we have seen, Grube came to his fundamental principle by the method of analogy; by this method, only probabilities and hypotheses can be established; it is seldom successful in making discoveries.

Especially does this method fail when applied to the subject of elementary arithmetic, as there is no essential resemblance between an object and a number in the question of origin and growth, and no resemblance whatever in the contents of these two ideas (the first three or four numbers making an exception as to their growth).

We have only to attempt to compare a number with a sensible object, in regard to the correlative psychological processes that go on in our minds, and the wide difference will be made manifest at once.

We perceive sensible objects as soon as they make an impression upon our mind, and we attend to these impressions and refer them back to the exciting cause. These sense impressions do not disappear with the removal of the exciting cause, but leave a trace in our mind in the form of an image, which recalls the object when that is not present before our consciousness.

In a word, all sensible objects are, using the terms of Herbert Spencer, "presentative representative." The case is different with numbers. Our psychological analysis has shown that only the first three or four numbers are presentative representative; but the perception of the remaining numbers up to the number 10 is not very clear, and their clearness is diminished as they increase in magnitude. Hence they can never be definitely perceived without counting.

But, as will be shown later, objective means still render here a service; hence we may regard numbers up to 10 as representative; as to numbers above 10, they are all mainly re-representative, all being conceived by relating them to other numbers that are representative in their nature. All work with numbers above 10 is performed upon symbols that offer no sensible suggestion whatever.

We may forget the name of an object and still have a picture of it; hence the name is not the essential element in the perception of a sensible object. But it is altogether different with a number; once we forget

its name the number itself disappears. This fact perfectly illustrates the re-representative process in the idea of numbers.

Further, let a child see a lion for the first time, and the perception of this animal will be sufficiently clear and definite to recognize it if seen again, provided there is nothing else in the field of consciousness to hinder it. The case is different with numbers. Let a skilful reckoner count a series of objects, say 13; will he be able to tell by mere sense-perception another group of 13 similar objects that are near-by without counting?

Indeed, if we could acquire the idea of numbers, within the limit of 1 to 100, in the same way as the first three numbers, by the mere process of sense-perception, all arithmetic within that limit would be utterly superfluous. What use would there be of mediate processes when immediate ones are at hand?

By comparing a number with a sensible object, Grube neglects one of the most important attributes of a number—one which goes to make up its essential content; this is the scale relation of number.

By mere sense-perception we apprehend a sum as a plural quantity; but to ascertain the number we must count, and thus establish a definite quantitative relation of the sum to a certain member of the scale; hence a number cannot be comprehended as a separate individual, or as a whole, by itself, but only as a member of a scale or series.

Indeed, if we could comprehend every number as an individual by itself, and not as a member of a series, then why should, in our instruction, number 5 follow 4, and not the reverse; why could not 7 follow 4, or 11 follow 8, just as, in object-lessons, the study of one object may either precede or follow the other?

The view that number can impress itself upon the mind as a sensible object has resulted in the too great neglect of counting; and what is still worse, is the fact that the acquired numbers do not stand in any relation to each other—not being derived one from another, they easily disappear from the mind.

Numerous examples could be cited from primary arithmetics, whose authors consider counting a superfluous process in the first year or two of instruction in numbers, but we shall have to limit ourselves to the

work of one author, Dr. E. E. White,* who, while telling us that he differs in some points most widely from the Grube Method, still holds fast to the fundamental principle of the latter.

He gives in his "Oral Lessons in Number" the following:

"Special attention is called in this connection (in connection with teaching number the first year) to the importance of avoiding in these primary processes the too common practice of counting by 1.

"The numbering, combining, and separating of groups of objects by counting leads to the pernicious habit of adding and subtracting numbers by counting—a habit that must be overcome before a pupil can learn to add and subtract numbers as *wholes.*"

Further: "The twofold aim of all exercises, in the first and second years, is to impart a clear idea of numbers from 1 to 20, and to give the pupil the power to add and subtract the primary or digital number without counting."

These conclusions, that have for a long time misdirected the instruction in primary arithmetic, are but the logical consequences of the application to our subject of the principle of sense-perception, without knowing its function and import, and hence, its limitation.

We too believe in the value of objective means for teaching primary work, in number, but we are also aware of its limitations.

We are able to make an estimate only of three or four objects without conscious counting; hence, objective means are of value in teaching number from 1 to 4; further, they can assist us in conceiving numbers from 4 to 10, because the series of numbers from 1 to 10 consists of sums or differences of two numbers, of which one can be developed without conscious counting, as, for instance, $7+2$, $8-2$, $5+3$, $4+4$, etc.; only two examples make an exception in this series: $5+5=10$, and $10-5=5$.

Further, the grouping of objects in certain geometrical forms facilitates our process of counting, such as dots on a domino, or spots on playing-cards. Hence there is a use of certain number-images, from 4 to 10, which may serve as an intermediate step between the spontaneous number and the symbolic number.

* E. E. White, "Oral Lessons in Number," New York and Cincinnati, 1884.

These number-images may also help us to prepare the way for the ntroduction of ciphers.

This grouping of objects or points facilitates the process of count- ng, but does not make the process superfluous; this can be proven by he fact that there are certain number-images that complicate and pro- ract the process of counting.

To this latter belongs the grouping of points in a circular form. ith one point as a centre, in the following way:

For the same reason we are not able to tell instantly the number of des of a regular polygon that has more than five sides.

The unfortunate use made by Grube of the principle of sense- erception in teaching number having been clearly shown, we can now ter into a more detailed criticism, in order to see how the principle of nse-perception, with the logical consequences that follow from it, has fluenced practice.

Grube gives the two following reasons why we should endeavor to ve each number, with all its constituent parts from 1 to 100, clear in e mind of the pupil:

First. Because, only the numbers from 1 to 100 are directly intui- ble.

Second. Because all reckoning with larger numbers can only be erformed by relating them to the first hundred.

Both reasons are false—the first, from the point of view of the sychology of number, as the first three or four numbers only are the irect result of sense-perception; the second, from the point of view the theory of arithmetic and the psychology of number; for we have en in the analysis of the growth of the idea of number that, at the end the second stage, the concept of number is assisted by the grouping rocess, which consists in regarding a certain aggregate as a unit and eriving all units of higher order from it; in a word, of forming a stem of numeration.

In our decimal system of numeration all numbers are related to 10 or to a multiple of 10. In this case 100 is to be regarded as a concept derived from 10, *i.e.*, $10 \times 10 = 100$.

The individual consideration of each number, for the purpose of making it clear to the mind, cannot be justified by the psychology of number, as this would destroy the essential characteristic of the idea of number, namely, its serial nature.

Further, the individual consideration with a view to making each number clear to the mind is superfluous above 10, as this is done by the decimal system of numeration.

In fact, every system of numeration has for its object the facilitation of the conception of numbers; and the more closely we adhere to a system the easier and clearer is our concept of number.

To convince ourselves of this, it is sufficient to represent some numbers by a system different from our own. For example, let us represent the numbers 8, 9, 10, 11, 12, 13, and 14 according to the septenary system, and we shall get 11, 12, 13, 14, 15, and 16. At first the concept of these numbers will be very vague, but they will become clearer the more we get used to this mode of representation.

According to our decimal system of numeration, all numbers from 10 to 100 are derived from the first 10 numbers; for instance, the number 36 shows us that its component parts are $10 + 10 + 10 + 6$, or $3 \times 10 + 6$; with this, our concept of the number 36 is sufficiently clear, and cannot become any clearer. As we have noticed before, the clearness of a concept of number is diminished as it increases in magnitude.

Does the child have a clearer concept of 36 when he considers it from the point of view of numeration, or when he considers it in the following manner: $35 + 1 = 36 = 30 + 6 = 31 + 5 = 32 + 4 = 33 + 3$, etc., without having the slightest idea of our system of numeration, and without knowing the meaning of addition and of the other following operations?

If a child knows how to form numbers above 10 according to the laws of our decimal system, and knows how to count up to 36, he also knows then how to count up to 39; hence there is no reason to break

the series at 36 and perform the four fundamental operations from 1 to 36. Such a procedure is very unnatural, and can be justified neither by psychology nor by the theory of arithmetic.

One who knows how to count up to 6 may not know how to count up to 9; but one who knows how to count up to 46 knows also how to count to 56.

As we have already noticed, a few pages back, that the extensive application of the principle of sense-perception results in the neglect of counting, so we must notice now that the individual consideration of each number above 10 results in not making any use of our decimal system of numeration.

The old writers of arithmetical text-books, who were mathematicians but not pedagogists, went to one extreme by beginning their instructions in arithmetic with the explanation of our system of numeration; Grube goes to the other extreme by teaching number to pupils, for two and a half years, without availing himself of our decimal system, and without pointing out its significance to his pupils.

We must add here that in the method of teaching number which we shall propose, every new number from 1 to 10 is, for pedagogical reasons to be stated later, introduced to the pupil by one kind of material, or by certain definite objects derived out of the sphere of his knowledge, and thus the distraction of attention by the variety of objects in the Grube Method is avoided.

This may at first sight give to our mode of presentation the character of the individual consideration of the Grube Method; but, in reality, the proposed method differs widely from that of Grube, as by it the serial nature of number is preserved; further, it does not confuse the pupil by practising the four fundamental operations upon each individual number.

Grube uses the four fundamental operations simultaneously, as a means of strengthening the perception of number; then, of course, it is supposed that the pupil knows how to perform these operations and knows their logical meaning; but where did he acquire that knowledge?

To use the four operations with the first lesson of instruction, as a

means of strengthening the percept of number in children, is to make an application of knowledge they do not possess.

The progress that a pupil makes, applying Grube's mode of presentation, is not in learning a new process but in learning a new object—a new number—by means of the four operations, which being used only as a means to an end, stand near each other instead of following each other, and being gradually developed one from another.

The knowledge of addition presupposes the knowledge of forward counting, of which addition is an abbreviated process; the knowledge of subtraction presupposes the knowledge of backward counting. Further, we ought to teach, first, addition and subtraction, and only later multiplication and division, as the last two operations involve a more general process and can only be developed from the former two operations.

The need for shorter processes must first be felt by the pupil before they are brought into use; such is the historical way, and also the natural way for the development of the process of multiplication. The same is also true of division. It seems to me then that a child will never clearly see the difference in the process of $2 + 2 = 4$ and $2 \times 2 = 4$, when taught simultaneously, but will always take one for the other.

Our psychological analysis of the idea of number has shown that generically every new number is formed by the successive addition of a unit to the next lower number in the scale; that is to say, we form numbers at first inductively; later, we form them in many ways, all of which are mainly deductive; whereas Grube begins his first step of teaching numbers deductively.

The separation and resolving of numbers into their equal and unequal parts, which is the first step of the set form of the Grube Method, is nothing else but a deductive application of a previous process of combination of units with which we ought to begin.

Instead of teaching the number 6 inductively, by counting sensible objects, Grube starts his lesson in teaching the number 6, by asking his pupils what they known of the number 6, and they begin to separate it into its component parts according to the set form.

Now this procedure is merely the following of a mechanical process which taxes the memory to the utmost; and at the same time, every conclusion of this process, if once forgotten by the pupil, cannot be reproduced again by him, as he possesses no means by which to do it.

The separation of a number into equal and unequal parts brings back continually the easy exercises; this results in an unnecessary accumulation of material; at the same time, the new and difficult exercises are not sufficiently emphasized, being treated with the old and easy material.

Among many pedagogical disadvantages of the Grube Method besides those mentioned above, we may point out the following:

(1) The form of instruction is exclusively the catechetical; according to this form the teacher gives the pupil the leading questions; the latter is merely a passive recipient, no room being left for the exercise of his self-activity. (2) The constant use of the same form of exercises and their endless repetition tire the pupil and weaken his interest for the subject.

Grube gives preference to abstract numbers, and the greater part of his exercises consists of these numbers; he assigns to problems (*i.e.* applied arithmetic) a secondary place, and they appear only after the all-sided study of the number.

Further, the material used in these problems has no interest for the pupil. The uniformity of his exercises trains the pupil to a sort of mechanism. It is sufficient for a pupil to know the first line of the setform, and he will write out the other lines mechanically; hence it is difficult, by this form of instruction, to excite in the pupil a love for the subject as claimed by Grube.

The extensive use of fractions at an early stage of instruction, as practised by Grube, is above the comprehension of a beginner, as *e.g.*, the question on page 35 of Grube's Manual, while teaching the number 5: "How many times 1 is a half of 6 more than a half of 4;" and "how much less than 5?" Or on page 48: "$\frac{1}{3}$ of 12 is what part of 8?"

Grube begins written work with his first lesson in number, to which undue prominence is given; hence he introduces ciphers and the sym-

bols of the four operations with his first lesson ; but really we have no use for these signs when dealing with spontaneously developed numbers. Therefore ciphers ought not to be introduced before we come to the number 5, perhaps not until later; and the symbols of the four operations ought to be introduced gradually, and not all at once, and at a later stage than ciphers.

Here ends our critical examination of the Grube Method; we have considered it from several points of view, and we state our conclusions as follows :

1. Grube's Method is false from the point of view of *arithmetic as a science and an art.* It ignores the process of counting. It does not teach operations, and does not avail itself of our decimal system of numeration. It substitutes the individual study of each number for the essential reality of the subject of arithmetic, which is the study of operations.

2. Grube's Method is false from the point of view of *psychology,* as we do not acquire the idea of number by the process of sense-perception (with the exception of the first three or four numbers); it ignores the serial nature of number and begins with deductions instead of inductions.

3. The Grube Method is false from the point of view of *pedagogy,* as with so many operations upon each number it confuses the child; it demands too great an amount of attention on the part of the pupil—an amount which he is utterly unable to give. The method destroys all the spontaneity of the pupil ; it taxes his memory to the utmost, as one step never leads up to the next.

4. The set form of the Grube Method is a mere mechanism ; the material of the subject-matter is made up of abstract numbers which cannot excite an interest in the child ; the separation of each number into equal and unequal parts is a tedious process and aimless in its results.

CHAPTER IV.

A METHOD OF TEACHING NUMBER FROM 1 TO 100.

OUR work would be only half done if we should limit ourselves to the mere exhibition of the falsity of the Grube Method, as no criticism can ever be effective and complete without showing how to build another structure in place of the one condemned; for this reason we shall devote a few pages to the main outlines of a method we would employ in teaching numbers from 1 to 100, as it is in the teaching of these numbers that the Grube Method has done the greatest harm.

We shall regard our task accomplished when we have answered these three questions:

1. What is to be taken as a guide for the arrangement of the subject-matter, and what is to be this arrangement?

2. What are the demands of pedagogy in teaching primary arithmetic in order to make the instruction educative?

3. What is to be the concrete form of a course of teaching number from 1 to 100, that it may harmonize with the several principles derived from psychology, pedagogy, and arithmetic as a science and as an art?

To answer the first question, we must take for granted:

First. That counting is primarily the essential process in forming numbers, and that the different ways of counting give rise to the four fundamental operations; hence we must begin our instruction of elementary arithmetic with counting, and gradually and systematically develop from it the fundamental operations.

Second. That all operations with large numbers are made possible, and are greatly simplified, by being reduced through the help of our system of numeration to the two fundamental tables—that of addition and multiplication; hence we are to consider each operation first within the limits of these tables.

Third. That a system of numeration expresses every number in the form of a sum, every addend of which (the units excepted) represents a product.

Further, in order to make use of a system of numeration in the performance of operations, we must know how to transfer a unit of a lower order to a higher order, and the reverse; hence, in order to understand and make use of a system of numeration, some knowledge of the four operations is necessary. (The above statements should be fully developed in every treatise that deals in a thorough manner with the method of teaching elementary arithmetic).

The above considerations justify the arrangement of the subject-matter of elementary arithmetic in three main divisions, each division having certain subdivisions, which are again subdivided into many method-wholes.

The three main divisions are:

1. Operations with numbers from 1 to 10.
2. Operations with numbers from 10 to 100.
3. Operations with numbers above 100.

In the First Division, our decimal system of numeration can render us no service; every operation in this division is necessarily reduced to counting. Further, the pupil ought to remember most of the results of the operations of this the First Division of the subject; memory drill is not necessary, as this ought to come naturally from sufficient practice in solving problems and working examples; otherwise in our next division, from 10 to 100, we should need again the assistance of counting with objects in teaching addition and subtraction, and we should also need to reduce every process of multiplication to an addition, and every process of division to a subtraction.

In the Second Division, where every number consists of two groups of units of different order,—of simple units, and units of the second order, tens,—we begin to use to some extent our decimal system; but in order that the pupil may be able to come into complete possession of it, while writing numbers and operating with them, he ought to be led up to the decimal system gradually.

The best way of doing this is by a series of exercises in numbers

composed only of tens, without any units; the numbers in these exercises should be treated simply as units of a higher order. These exercises are to serve as a transition from units of the first order to numbers consisting of units of the first and second order.

According to the above considerations, this division is to have a subdivision consisting of exercises with numbers composed only of units of the second order. (An almost similar subdivision of exercises composed of units of the second order only we find in Hentschel's "Lehrbuch des Rechenunterrichts," the first edition of which was published in 1842).

An exposition of the Third Division and its arrangement of the subject-matter is not within our present purpose; still, we may remark that when a pupil has once been led up to 100 by our order of the subject-matter, he can afterwards follow more or less the arrangement of the existing text-books.

We shall now attempt to answer the second question—"What are the demands of pedagogy in teaching arithmetic?"

Pedagogy demands that we should educate through instruction; *i.e.*, we are to create an interest for the ideas that are to be introduced to the youthful mind, and thus gain the coöperation of the pupil for the reception of the material of the subject-matter.

By what means are we to gain the interest of the pupil for our subject?

An answer to this question can be found in the principles of instruction formulated by the Herbartian School, and known by the name of concentration.

These principles when properly understood and fully carried out are destined to become the presiding and influencing agents in the construction of our courses of study, and to infuse new light into the method and subject-matter of instruction; and it is only by fully carrying out these principles that we can ever hope to shake off the present routine in which the art of instruction is deeply plunged.

These principles call our attention to the fact that the historical priority of mathematical studies resulted in their study, independently of other branches. This is the cause of our utter lack of interest in the abstract division of mathematics (to this division belong arithmetic

algebra, and calculus), which is nothing other than a method of reasoning, and which can never gain our highest interest unless studied in connection with the matter to which it is applied.*

The principles of concentration, which are of late beginning to receive due consideration, reveal the dryness of the instruction as at present given in the subject of arithmetic and the want of interest in it on the part of the pupil, and point out the cause of this in the present isolation of arithmetic from other branches of study, which is due to the fact of beginning instruction with abstract number.

Even the applied examples of most of our text-books excite very little interest in the pupil, as the material of these problems is mostly taken from a sphere outside the experience of the pupil.

The above-mentioned principles, together with the criticism of the present method of teaching elementary arithmetic, point out the way of construction, which consists in making the synthetical method of instruction prevail, instead of the present analytical view of the mere mathematician.

Under the dominance of the view of the pedagogist whose function is to harmonize the interest of the pupil with the special end of each branch of study, the entire course of arithmetic is to be constructed upon a graded, well-selected, and systematized series of simple problems, as only by means of problems (not examples) can we explain the need and meaning of operations upon numbers.

We are to begin every method whole, not with abstract examples, but with applied problems to which we are to return again at the end of each new step. Further, the material of the problems ought to be taken out of the sphere of the pupils' experience, and thus, by means of the material itself gain the interest of the pupil for the special scientific end of our subject.

(While writing these lines a recently published book, under the title "Number Work in Nature Study," by W. S. Jackman, advocates in its introduction something of the method for which we are pleading; but his work cannot be regarded as an attempt at carrying out the pedagogical principles of concentration and coördina-

* The fundamental division of mathematics into abstract and concrete, with illustrations of their different objects and natures, is fully explained in the "Philosophy of Mathematics" of Auguste Comte.

tion of the instruction in this subject, but rather as an attempt to show how natural science can be benefited by the introduction of the quantitative element. A book that could satisfy our demand would have to show how to carry out the pedagogical principles without neglecting in the least the nature of our subject or the arrangement of the subject-matter.)

COURSE FROM 1 TO 10.

This course consists of seventeen steps or method-wholes; the first eight steps are forward and backward counting by successive addition and subtraction of a unit.

First	step	.	.	.	.	.	.	.	.	counting	from	1	to	3
Second	"	.	.	.	.	.	.	.	.	"	"	1	to	4
Third	"	.	.	.	.	.	.	.	.	"	"	1	to	5
Fourth	"	.	.	.	.	.	.	.	.	"	"	1	to	6
Fifth	"	.	.	.	.	.	.	.	.	"	"	1	to	7
Sixth	"	.	.	.	.	.	.	.	.	"	"	1	to	8
Seventh	"	.	.	.	.	.	.	.	.	"	"	1	to	9
Eighth	"	.	.	.	.	.	.	.	.	"	"	1	to	10

Every step is to be subdivided into three parts:

First. The introduction of each number to the pupil by a special object out of the sphere of the pupil's knowledge, in order to gain the interest of the pupil for the step.

For instance, we are to begin the third step by counting the fingers of one hand; the fifth step, by counting the leaflets of the leaf of a horse-chestnut tree.

Second. By drawing squares and counting them successively one after the other; then crossing off one after another while counting them backwards, or by using cubical blocks grouped according to the purpose of each step.

We consider the drawing of rows of squares by the pupils as one of the best objective means to be *used in the instruction of primary arithmetic;* especially when they are used in connection with Tillich's calculating-box—an apparatus worthy of wider adoption than it has yet received in this country.*

* In a work which we have projected, we purpose to present a full discussion of

The squares are within the possession of the pupils; they are easily understood. Further, by means of these squares we can best illustrate the serial nature of numbers; we can also illustrate the complex units of the second order, and the fundamental operations. Their use also brings into play the motor element.

Third. Applied examples of forward and backward counting.

The task of solving the question of the selection of apparatus is not an easy one, when there are over a hundred kinds in Germany alone, each one offering some advantages.

The mode of presentation of any of these steps, for instance the fifth, may be the following:

FIFTH STEP.

Let the pupils count the leaflets of the leaf of a horse-chestnut tree.

Let them pluck off one leaflet at a time and count backwards.

Let the pupils draw a square for every leaflet of the leaf and count the squares.

Let them cross off one square at a time and count backwards.

Let the pupils count the days of the week (Sunday one, Monday two, Tuesday three, etc.).

How many flowers are on my desk?

How many flowers remain if I throw away this one?

(In the proper season of the year the pupils may be asked to count the stamens and the parts of the calyx of the star-flower, *Trientalis.*)

I had seven marbles and lost one. How many marbles had I left? Afterwards I lost one marble. How many marbles remain?

I had seven cents and spent one cent for a pencil and one cent for a paper. How many cents were left?

How old are you? Six.

How old will you be in a year from now?

Five and one are how many?

Six and one are how many? etc., etc., etc.

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the question where and how far objective means can render service in the study of elementary arithmetic, and also of the question of the selection of apparatus for objective illustrations, that can be justified by psychology and by the science and art of arithmetic.

All the work with numbers from 1 to 10 can be performed orally; written work can only indicate the operations and their results, but cannot facilitate the way of acquiring a result; hence there is no need of haste in beginning written work with ciphers.

The time of the introduction of ciphers is to depend upon the three following considerations: First. Upon the organization of the school; that is to say, there may be a need of introducing written work earlier, in order to keep the pupils employed when the teacher has several grades in one class. Second. Upon the general preparation of the pupils for written work. Third. Upon the amount of arithmetical knowledge the pupils possess, in order that the written work may help them in their further progress in arithmetic.

Under no consideration can written work with ciphers be justified if introduced before the eighth step. Ciphers may be introduced as abbreviations of the squares of the previous steps.

The introduction of ciphers forms the Ninth Step.

The Tenth Step consists in adding to a number or in taking away a definite group of units. (Addition and subtraction.)

The Eleventh Step.—Solving problems and working examples, involving addition and subtraction of a more complex nature than the previous step.

The exercises of this step are of the following nature:

1st. $4 + 3 + 2 =$

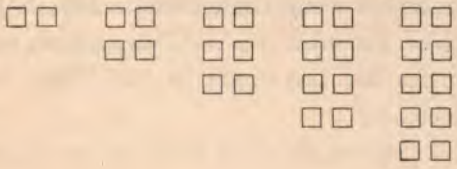
2d. $8 - 3 - 4 =$

3d. $9 - 2 + 3 - 4 =$

The Twelfth Step.—Forward counting with twos. (Multiplication by two.)

Counting with twos is to be conducted as a natural continuation of the preceding exercise.

This step may be introduced in the following form:



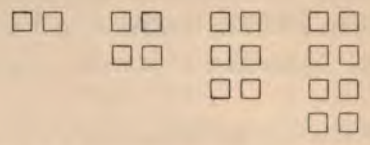
$2+2=$; $2+2+2=$; $2+2+2+2=$; $2+2+2+2+2=$; then multiplication by two ought to be developed by means of problems which are to be solved, first, by addition, and afterwards by multiplication. These problems, besides showing the pupils the need for successive addition of twos, ought to make them aware of the number of times two is taken.

After the pupils have solved many problems, where two is repeated two, three, four, and five times, the teacher is to acquaint the pupils with the sign of multiplication, showing them that instead of $2+2+2+2=8$, we write shorter $2 \times 4 = 8$, and the cross ($\times$) stands for the word repeat.

This step is to be completed with problems and examples involving the knowledge of all previous steps.

The Thirteenth Step.—Backward counting with twos. (Division by two.)

The pupil may be led up to this step in the following form, crossing off two squares after each subtraction:



$2-2=$; $4-2=$; $6-2=$; $8-2=$; $10-2=$.

$4-2-2=$; $6-2-2-2=$; $8-2-2-2-2=$; $10-2-2-2-2-2=$. Then by means of problems of the following nature we are to teach the need and meaning of division.

Problem.—A boy made note-books out of eight sheets of paper; he used two sheets for each note-book. How many note-books did he make?

Solution.—When a boy has made one note-book, he has $8-2=6$ sheets left; when he has made the second note-book, he has $6-2=4$ sheets left; when he has made the third note-book, he has $4-2=2$ sheets left; with the last two sheets he makes the fourth note-book. Answer, four note-books.

In solving this problem the pupil must be led to see that to get the

answer he had to subtract twos from 8. Further, the pupil must notice that the number of times he had to subtract is the same as the number of note-books. By solving similar problems the pupil will learn that the answer in each problem depends upon the number of times two can be subtracted from the given number, and upon the number of times two is contained (repeats itself) in the given number.

The Fourteenth Step.—Forward and backward counting by threes. (Multiplication and division by three.)

The Fifteenth Step.—Forward and backward counting by fours. (Multiplication and division by four.)

The Sixteenth Step.—Forward and backward counting by fives. (Multiplication and division by five.)

The Seventeenth Step.—Problems and examples involving the four operations.

Every step beginning with the Tenth Step is to be subdivided into three parts: (1) Performing operations with the assistance of squares that are to be drawn by the pupil and grouped according to the purpose of each step; (2) Solving problems arranged for the purpose of each step; (3) Working examples.

During the Tenth Step, the symbols of the operations of addition and subtraction are to be introduced.

During the Twelfth Step, the symbol of multiplication is to be introduced, and during the Thirteenth Step, the symbol of division.

The work of the Tenth Step may be regarded as completed when the pupils are able to find every one of the following sums and every one of the following differences:

$1+2$	$1+3$	$1+4$	$1+5$	$1+6$	$1+7$	$1+8$	$1+9$
$2+2$	$2+3$	$2+4$	$2+5$	$2+6$	$2+7$	$2+8$	
$3+2$	$3+3$	$3+4$	$3+5$	$3+6$	$3+7$		
$4+2$	$4+3$	$4+4$	$4+5$	$4+6$			
$5+2$	$5+3$	$5+4$	$5+5$				
$6+2$	$6+3$	$6+4$					
$7+2$	$7+3$						
$8+2$							

First. When the pupils are thoroughly acquainted with the idea of regarding tens as complex units of counting.

Second. When the pupils are familiar with the representation of these complex units in writing.

Third. When the pupils see clearly the analogy between the units of the second order and the units of the first, and are able to reduce all problems and examples with units of the second order to the operations with units of the first order.

Counting with numbers composed of units and tens from 1 to 100.

Counting with numbers composed of units and tens is one of the most important divisions of the whole course of elementary arithmetic.

Here the operations with numbers offer an opportunity to make use of our system of numeration, and of utilizing certain results of the previous division.

Further, the operations of this division contain most of the results which form the basis of all arithmetical tables that make operations with large numbers possible.

And last, in this division, the pupils become acquainted, while performing the operations, with the advantage of the decimal system of numeration. Hence their work in the third division differs from the elementary ways of the first division.

It is true that all the operations with numbers, even in this third division, could be reduced to the operations with numbers as in the first division ; but the reasons why this should not be done are very obvious.

This third division consists of several subdivisions. The first is the formation of numbers from 1 to 100, of units and tens; the object of this subdivision is to teach the pupils to determine the number of units and tens. The exercises must be performed, first, by the help of the abacus, or by means of drawing rows of squares, and then without any objective means.

The main steps of this subdivision are:

First. The successive addition of units to tens, and the successive subtraction of units from tens.

Second. Addition of groups of units to tens, and the successive groups of units from tens.

Third. Successive addition of tens to numbers composed of uni and tens; successive subtraction of tens from numbers composed of unit and tens.

The second subdivision is addition and subtraction; with the beginning of this division there ought to be no further need of any objective means for facilitating the processes.

This second subdivision consists of the following steps :

First. Addition and subtraction of groups of units from number composed of tens only.

Second. Addition of numbers whose sum of units does not exceed ten, and subtracting numbers, when the units of the subtrahend are less than the units of the minuend.

Third. Addition, when the sum of the units equals ten; and subtraction, when the minuend consists of tens only.

Fourth. Addition, when the sum of units exceeds ten; subtraction when the units of the minuend are less than the units of the subtrahend.

All the exercises of this and the following subdivision consist of problems and examples.

The object of the third subdivision is to teach multiplication and division, and consists of the following steps:

1. Multiplication and division by..... 2
2. " " " " 3
3. " " " " 4
4. " " " " 5
5. " " " " 6
6. " " " " 7
7. " " " " 8
8. " " " " 9

Every step is to be introduced to the pupil by forward and backward counting with equal groups of units.

Ninth step. Multiplication with a multiplicand of two digits.

Tenth step. Division with a divisor of two digits.

Eleventh step. Elementary ideas of fractions.